Exercise 7I: Solutions

1 a

$$\begin{aligned}\phi - 1 &= \frac{1+\sqrt{5}}{2} - 1 \\ &= \frac{1+\sqrt{5}-2}{2} \\ &= \frac{\sqrt{5}-1}{2}\end{aligned}$$

$$\therefore \quad \frac{1}{\phi} = \phi - 1$$

b

$$\begin{aligned}\phi^3 &= \frac{(1+\sqrt{5})^2(1+\sqrt{5})}{8} \\ &= \frac{(1+2\sqrt{5}+5)(1+\sqrt{5})}{8} \\ &= \frac{(6+2\sqrt{5})(1+\sqrt{5})}{8} \\ &= \frac{6+8\sqrt{5}+10}{8} \\ &= \frac{16+8\sqrt{5}}{8} = 2+\sqrt{5}\end{aligned}$$

$$\begin{aligned}2\phi + 1 &= 1+\sqrt{5}+1 \\ &= 2+\sqrt{5}\end{aligned}$$

$$\therefore \quad \phi^3 = 2\phi + 1$$

c As shown above, $\phi - 1 = \frac{1}{\phi}$.

$$\therefore \quad (\phi-1)^2 = \frac{1}{\phi^2}$$

$$\begin{aligned}2 - \phi &= 2 - \frac{1+\sqrt{5}}{2} \\ &= \frac{4-1-\sqrt{5}}{2} \\ &= \frac{3-\sqrt{5}}{2}\end{aligned}$$

$$\begin{aligned}(\phi-1)^2 &= \left(\frac{1+\sqrt{5}-2}{2}\right)^2 \\ &= \frac{(\sqrt{5}-1)^2}{4} \\ &= \frac{5-2\sqrt{5}+1}{4} \\ &= \frac{3-\sqrt{5}}{2} = 2-\phi\end{aligned}$$

$$\therefore \quad 2-\phi = (\phi-1)^2 = \frac{1}{\phi^2}$$

2 a In $\triangle ACX, \angle ACX = 90^\circ - \angle BCX$

In $\triangle CBX, \quad \angle B = 90^\circ - \angle BCX$

$$\begin{aligned}\angle ACX &= \angle B \\ \angle A &= \angle BCX \\ \triangle ACX &\sim \triangle CBX\end{aligned}$$

$$\therefore \quad \frac{AX}{CX} = \frac{CX}{BX}$$

b Multiply both sides of the above equation by $CX \times BX$

i $CX^2 = AX \times BX$

$$= 2 \times 8 = 16$$
$$CX = 4$$

ii $CX^2 = AX \times BX$
$$= 1 \times 10 = 10$$
$$CX = \sqrt{10}$$

3 Join AB and BC. This will produce a right-angled triangle with an altitude. In **Q** 2 we proved that the altitude was the geometric mean of the two segments that divided the base. Therefore, as in **Q** 2:

$$\frac{AD}{BD} = \frac{BD}{CD}$$
$$\frac{EC}{DE} = \frac{DE}{DE + EC}$$

Since $BD = DE$,

$$AD = EC \text{ and } CD = DE + EC$$
$$\frac{DE}{EC} = \frac{DE + EC}{DE}$$
$$= 1 + \frac{EC}{DE}$$
$$x = \frac{DE}{EC}$$
$$= 1 + \frac{1}{x}$$
$$\therefore \quad x^2 - x - 1 = 0$$

Using the quadratic formula:

$$x = \frac{-1 + \sqrt{1 - 4 \times 1 \times -1}}{2}$$
$$= \frac{-1 + \sqrt{5}}{2} = \phi$$

(Rejecting the negative root as $x > 0$)

$$\frac{EC}{DE} = \frac{1}{\phi} = \phi - 1$$
$$\frac{AD}{BD} = \frac{EC}{DE} = \phi - 1$$
$$\therefore \quad \frac{AD}{BD} = \frac{BD}{CD}$$
$$= \phi - 1$$

4 a a $\angle AOB = \dfrac{360}{10} = 36^\circ$

b $\angle OAB = \dfrac{180 - 36}{2}$
$$= 72^\circ$$

b a $\angle XAB = \dfrac{72}{2} = 36^\circ$
$$\angle ABO = \angle OAB = 72^\circ$$
$$\angle AXB = 180 - 36 - 72$$
$$= 72^\circ$$
$$\angle ABO = \angle AXB$$
$$\therefore \quad AX = AB$$

b $\angle XAO = \dfrac{72}{2}$
$$= 36^\circ = \angle AOX$$
$$\therefore \quad AX = OX$$

c Corresponding angles are equal, so the triangles must be similar.

c $\triangle AOB \sim \triangle XAB$

$$\frac{OB}{AB} = \frac{AB}{XB}$$

$$\frac{OX+XB}{AB} = \frac{AB}{XB}$$

$$OX = XA = AB$$

$$\frac{AB+XB}{AB} = \frac{AB}{XB}$$

$$1 + \frac{XB}{AB} = \frac{AB}{XB}$$

$$x = \frac{XB}{AB}$$

$$= 1 + \frac{1}{x}$$

$$\therefore \quad x^2 - x - 1 = 0$$

Using the quadratic formula:

$$x = \frac{-1+\sqrt{1-4\times 1\times -1}}{2}$$

$$= \frac{-1+\sqrt{5}}{2} = \phi$$

(Rejecting the negative root as $x > 0$)

$$\frac{XB}{AB} = \frac{1}{\phi}$$

$$= \phi - 1$$

$$= \frac{-1+\sqrt{5}}{2}$$

(Refer to **Q**1 part **a**.)

$$\frac{XB}{AB} = \frac{AB}{OB}$$

$$= AB$$

$$= \phi - 1 \text{ since } OB = 1$$

$$AB = \frac{-1+\sqrt{5}}{2} \approx 0.62$$

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$$\phi^{\circ} = 1$$

$$\phi^1 = \phi = \frac{1+\sqrt{5}}{2}$$

$$\phi^{-1} = \frac{1}{\phi}$$

$$\therefore \quad \phi = \frac{1}{\phi} + 1$$

$$\phi^2 = \phi\left(\frac{1}{\phi} + 1\right)$$

$$= 1 + \phi = \frac{3+\sqrt{5}}{2}$$

$$\phi^3 = \phi(1+\phi)$$

$$= \phi^2 + \phi$$

$$= (1+\phi) + \phi$$

$$= 1 + 2\phi$$

$$= \frac{4+2\sqrt{5}}{2} = 2 + \sqrt{5}$$

$$\phi^4 = \phi(1+2\phi)$$

$$= \phi + 2\phi^2$$
$$= \phi + 2(1+\phi)$$
$$= 2 + 3\phi$$
$$= \frac{4 + 3(1+\sqrt{5})}{2} = \frac{7+3\sqrt{5}}{2}$$

$$\phi^{-1} = \frac{1}{\phi}$$
$$= \phi - 1$$
$$= \frac{1+\sqrt{5}-2}{2} = \frac{-1+\sqrt{5}}{2}$$

$$\phi^{-2} = \frac{1}{\phi}(\phi - 1)$$
$$= 1 - (\phi - 1)$$
$$= 2 - \phi$$
$$= \frac{4-(1+\sqrt{5})}{2} = \frac{3-\sqrt{5}}{2}$$

$$\phi^{-3} = \frac{1}{\phi}(2-\phi)$$
$$= 2\left(\frac{1}{\phi}\right) - 1$$
$$= 2(\phi - 1) - 1$$
$$= 2\phi - 3$$
$$= \frac{2+2\sqrt{5}-6}{2} = \sqrt{5} - 2$$

$$\phi^{-4} = \frac{1}{\phi}(2\phi - 3)$$
$$= 2 - \frac{3}{\phi}$$
$$= 2 - 3(\phi - 1)$$
$$= 5 - 3\phi$$
$$= \frac{10-3-3\sqrt{5}}{2} = \frac{7-3\sqrt{5}}{2}$$

Alternatively, the surd expressions can be multiplied and simplified, for the same answers:

$$\phi^{-1} = \frac{1}{\phi}$$
$$\phi = 1 + \frac{1}{\phi}$$
$$\phi^{n+1} = \phi \times \phi^n$$
$$= \left(1 + \frac{1}{\phi}\right) \times \phi^b$$
$$= \phi^n + \phi^{n-1}$$

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$$t_n > t_{n-1}$$
$$\frac{t_{n+1}}{t_n} = 1 + \frac{t_{n-1}}{t_n}$$

Since the Fibonacci sequence is increasing, $1 < \frac{t_{n+1}}{t_n} < 2$.

This means the sequence is not diverging to infinity, and has a limit between 1 and 2.

If there is a limit, then when n is large,

$$
\begin{aligned}
\frac{t_{n+1}}{t_n} &\approx \frac{t_{n-1}}{t_n} \\
&= 1 + \frac{t_{n-1}}{t_n} \\
&= 1 + \frac{1}{\frac{t_{n-1}}{t_n}}
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{t_{n+1}}{t_n} \\
&\approx \frac{t_{n-1}}{t_n} \\
&= 1 + \frac{1}{x}
\end{aligned}
$$

$$\therefore \quad x^2 - x - 1 = 0$$

Using the quadratic formula:

$$
\begin{aligned}
x &= \frac{-1 + \sqrt{1 - 4 \times 1 \times -1}}{2} \\
&= \frac{-1 + \sqrt{5}}{2} = \phi
\end{aligned}
$$

(Rejecting the negative root as $x > 0$.)

Thus the sequence will approach ϕ as $n \to \infty$.